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$SO(3)$-INSTANTONS ON $L(p, q) \times \mathbf{R}$

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1. Introduction

The introduction of gauge theory as a tool for studying low dimensional topology has dramatically increased our understanding of three- and four-dimensional phenomena. The general philosophy is to relate the topology of the underlying manifold to the topology of a moduli space of anti-self-dual connections on a suitably chosen bundle. However, the global topology of these moduli spaces is generally difficult to extract; in fact, even the existence of instantons can be a hard question to answer.

In this paper, we study the existence of anti-self-dual $SO(3)$-connections with finite Yang-Mills action ($SO(3)$-instantons) over the manifolds $L(p, q)\times\mathbf{R}$, where $L(p, q)$ is a Lens space. Our result gives necessary and sufficient conditions for a bundle to support an instanton in terms of the Pontrjagin charge and the asymptotic data. The key to this result is a computation of the equivariant index of the Dirac operator on S^4 twisted by an anti-self-dual $SU(2)$-connection. One finds instantons on $L(p, q) \times \mathbf{R}$ if and only if this character, which is a priori a virtual character, is an actual one.

In addition, we show how to describe the global structure of the moduli space by a set of equations and explicitly determine the moduli spaces for a few examples. More specifically, we obtain a reduced moduli space by quotienting out the natural $\mathbf{R}$ action given by translation. One sees that the dimension of this reduced moduli space is always even. Furthermore, in the case when this dimension is zero, the reduced moduli space is a singleton, and when this dimension is two, it must be $S^2 - B$, where B is a collection of either 0, 1, or 2 points. One can presumably show that the reduced moduli space has a natural complex Kähler structure.

The original motivation for this work lies in the study of θ_3^H, the integral homology cobordism group of homology 3-spheres, and the compactness properties of orbifold moduli spaces (see [11], [10], and [16]). However, one may also view this paper as a first step to extending Floer's

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homology theory for integral homology 3-spheres [12] to rational homology spheres. Moreover, the geometric situation closely resembles that suggested by 't Hooft for the study of quark confinement [15].

The exposition is organized as follows. In §2, we will reformulate $SO(3)$-instantons on $L(p, q)\times\mathbf{R}$ as $SU(2)$-instantons on S^4 invariant under a cyclic group action. By employing the Penrose twistor construction, the question about ASD connections on S^4 is equivalent to finding certain holomorphic bundles over $\mathbf{CP}^3$. This is the essence of the ADHM classification of $SU(2)$-instantons and the subject of §3. All such instantons are realized as linear maps between two finite-dimensional vector spaces, $A(z)\colon W \to V$. One of the crucial steps, in both the ADHM classification and our construction here, is the interpretation of W and V as the kernels of twisted Dirac operators over S^4. We will see that the invariant instantons are described by equivariant linear maps.

An index computation forms the core of §4. Here we assume that there is an invariant instanton on S^4 for an action of $\mathbf{Z}/2p\mathbf{Z}$ on an $SU(2)$-bundle and determine W and V as $\mathbf{Z}/2p\mathbf{Z}$ representations through the equivariant index theorem. In §5, this leads naturally to obstructions to the existence of invariant instantons. It is shown that for an invariant instanton to exist the action must be built from $\mathbf{Z}/2p\mathbf{Z}$ actions that extend to $\widetilde{S^1}$ actions, where $\widetilde{S^1}$ denotes the double cover of S^1. The main result is stated in §5.1. §6 demonstrates the existence of invariant instantons for all of the actions built in this fashion. This completes the classification of actions on $SU(2)$-bundles over S^4 that fix an instanton and therefore classifies instantons on $L(p, q)\times\mathbf{R}$. §7 is devoted to deriving equations for the moduli space and then the determination of the moduli space for several examples.

2. Group actions

This section translates the problem of finding anti-self-dual $SO(3)$-connections on $L(p, q)\times\mathbf{R}$ into that of finding anti-self-dual $SU(2)$-connections on S^4 invariant under a cyclic group action. We first show that a group action on S^4 induces an action on the space of gauge equivalence classes of connections. A fixed point under this action gives a lift of the action on S^4 to the bundle over S^4, and this lifted action fixes a connection. This presentation is designed to fix conventions as well as provide background.

2.1. The $\mathbf{Z}/p\mathbf{Z}$ action on S^4. We take $L(p, q)$ to have the standard metric and orientation induced as a quotient of S^3. The universal cover of $L(p, q) \times \mathbf{R}$ is conformally equivalent to $\mathbf{C}^2 - \{0\}$, where the deck transformation group, $\mathbf{Z}/p\mathbf{Z}$, is generated by

$$\xi\,(x_1, x_2) = \left(\xi x_1, \xi^q x_2\right); \tag{2.1}$$

where $\xi = e^{2\pi\sqrt{-1}/p}$ is a primitive p root of unity, and q is coprime to p. Identify $\mathbf{C}^2$ with $\mathbf{H}$, the one-dimensional quaternionic space, by the isomorphism $(z_1, z_2) \mapsto z_1 + z_2 j$. Then the $\mathbf{Z}/p\mathbf{Z}$ action in (2.1), in terms of quaternionic multiplication, is written

$$\xi\,(x) = \zeta^{1+q} x \zeta^{1-q},$$

where $\zeta = e^{\pi\sqrt{-1}/p}$. This extends to an action of $\mathbf{Z}/p\mathbf{Z}$ on S^4 with two fixed points, denoted 0 and ∞. With its canonical metric and orientation, S^4 may be taken as the $\mathbf{Z}/p\mathbf{Z}$-equivariant conformal compactification of $\mathbf{C}^2 - \{0\}$.

Suppose that there is an ASD $SO(3)$-connection on $L(p, q) \times \mathbf{R}$ with finite Yang-Mills action. Pull the $SO(3)$-bundle and connection back to $\mathbf{C}^2 - \{0\}$ and extend it, using Uhlenbeck's removability of singularities result [18], to an ASD connection on an $SO(3)$-bundle over S^4. Since w_2 of an $SO(3)$-bundle over S^4 vanishes, lift the $SO(3)$-bundle and connection to an $SU(2)$-bundle and ASD connection over S^4. Thus, an $SO(3)$-instanton on $L(p, q) \times \mathbf{R}$ gives an $SU(2)$-instanton on S^4.

2.2. Actions on connections. We will review some basic facts about group actions on connections following the notation of [6]. Let M be a closed, oriented Riemannian 4-manifold and Γ a subgroup of $\mathrm{Diff}(M)$, the group of diffeomorphisms of M. Let P be a principal G-bundle over M. Throughout this section, we assume that

$$\gamma^*(P) \cong P \quad \text{for every } \gamma \in \Gamma. \tag{2.2}$$

First, we will show that there is an induced action on $\mathscr{B}$, the space of gauge equivalence classes of G-connections.

We let $\mathscr{G}$ be the gauge group of P and $\mathscr{H}$ be the group of bundle automorphisms of P covering an element of Γ on M. Because of (2.2), the following sequence is exact:

$$1 \to \mathscr{G} \to \mathscr{H} \xrightarrow{r} \Gamma \to 1,$$

where r takes a bundle isomorphism into the induced diffeomorphism on M. Let $\mathscr{A}$ be the space of all G-connections on P so that $\mathscr{B} = \mathscr{A}/\mathscr{G}$.

Then $\mathscr{H}$ acts on $\mathscr{A}$ by $h(\nabla) = h\nabla h^{-1}$. This action descends to an action of $\mathscr{H}$ on $\mathscr{B}$.

Let $\mathscr{G}_\nabla$ be the stabilizer of ∇ and $\mathscr{H}_\nabla$ be the subgroup of $\mathscr{H}$ that fixes ∇. Suppose that $[\nabla] \in \mathscr{B}$ is a fixed point under the action of Γ. Then the following sequence is exact:

$$1 \to \mathscr{G}_\nabla \to \mathscr{H}_\nabla \to \Gamma \to 1.$$

We now specialize to our case of interest: $\Gamma = \mathbf{Z}/p\mathbf{Z}$, $G = SU(2)$, and $M = S^4$. Because an $SU(2)$-bundle over S^4 is classified by its second Chern class and $\mathbf{Z}/p\mathbf{Z}$ acts on S^4 by orientation-preserving diffeomorphisms, assumption (2.2) is satisfied. Furthermore, every connection ∇ on a nontrivial bundle has stabilizer $\mathscr{G}_\nabla = \{\pm 1\} \cong \mathbf{Z}/2\mathbf{Z}$ so that the following sequence is exact:

$$1 \to \mathbf{Z}/2\mathbf{Z} \to \mathscr{H}_\nabla \to \mathbf{Z}/p\mathbf{Z} \to 1.$$

This shows that $\mathscr{H}_\nabla \cong \mathbf{Z}/2p\mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Hence, it is impossible, in general, to conclude that the $\mathbf{Z}/p\mathbf{Z}$ action on S^4 lifts to the bundle. However, there is always a $\mathbf{Z}/2p\mathbf{Z}$ action on the bundle that covers the $\mathbf{Z}/p\mathbf{Z}$ action on S^4 and fixes ∇.

Notice that $\mathbf{Z}/p\mathbf{Z}$ acts on S^4 by orientation-preserving isometries so that the action on $\mathscr{B}$ restricts to an action on $\mathscr{M}$, the moduli space of ASD connections on P. Suppose then that ∇ is an ASD $SU(2)$-connection on S^4 obtained from $L(p, q) \times \mathbf{R}$. Then $[\nabla]$ will be invariant under the $\mathbf{Z}/p\mathbf{Z}$ action on $\mathscr{M}$ and hence defines a $\mathbf{Z}/2p\mathbf{Z}$ action on the $SU(2)$-bundle fixing ∇. Conversely, an invariant connection descends to a connection on $L(p, q) \times \mathbf{R}$. Then the problem of finding ASD $SO(3)$-connections on $L(p, q) \times \mathbf{R}$ is equivalent to finding $\mathbf{Z}/2p\mathbf{Z}$-equivariant $SU(2)$-bundles over S^4 whose action fixes an ASD connection.

2.3. Characterizing $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundles. We will develop a convenient language in which to discuss $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundles. Up to a gauge transformation, an action of $\mathbf{Z}/2p\mathbf{Z}$ on the $SU(2)$-bundle $P \to S^4$ may be described by the weights of the action on the fixed fibers over 0 and ∞ (see [10]). Using $\zeta = e^{\pi\sqrt{-1}/p}$ as the generator of $\mathbf{Z}/2p\mathbf{Z}$, denote these weights by m and m', respectively. A $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle will then be denoted by (k, m, m'), where k is the second Chern class of the bundle and m and m' describe the weights of the $\mathbf{Z}/2p\mathbf{Z}$ action.

Suppose that a connection ∇ is fixed by the action described by (m, m'). Now $\zeta^p = -1$ acts as the identity on S^4 so it acts as a gauge transformation on the bundle; this gauge transformation is in the stabilizer of ∇ and hence equal to ± 1. Now the action over 0 is given as $(-1)^m$ and over ∞ as $(-1)^{m'}$ and hence it is necessary that

$$m \equiv m' \mod 2. \tag{2.3}$$

We will assume this for the remainder of the paper.

Regard the following actions as equivalent:

$$\left(m, m'\right) \sim \left(-m, m'\right) \sim \left(m, -m'\right) \sim \left(p+m, p+m'\right). \tag{2.4}$$

The first three actions are all gauge equivalent while the fourth is obtained from the first by multiplying by the gauge transformation -1. If any one of these actions fixes a connection, all of the equivalent actions must fix a gauge equivalent connection.

Given an action, it will be useful to express it as $(m, m') \sim (aq-b, aq+b)$, where a and b are integers. To do this, we must find solutions to the equations

$$\begin{aligned} 2aq &\equiv m' + m \mod 2p, \\ 2b &\equiv m' - m \mod 2p. \end{aligned} \tag{2.5}$$

Solutions a, b will always exist since q is coprime to p and m, m' have the same parity. We call the pair a, b a solution if it solves (2.5) for some equivalent bundle in (2.4).

We will see later in §4.2 that $ab \equiv k \mod p$, where $k = c_2(P)$. This also follows from [10] and should be viewed as as topological restriction on the $\mathbf{Z}/2p\mathbf{Z}$ actions on P arising from a computation of the equivariant Chern character; that is, it is necessary that this condition be satisfied for the existence of the equivariant transition function $S^3 \to SU(2)$. Moreover, given a pair of integers a, b such that $ab \equiv k$, there is a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle $(k, aq-b, aq+b)$. We will see that there are, in addition, analytical obstructions to the existence of invariant ASD connections for a given $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle.

3. $SU(2)$-instantons on S^4

In this section, we present the ADHM classification of $SU(2)$-instantons on S^4. The essential point here is the reinterpretation of the anti-self-duality equations on S^4 in terms of holomorphic bundles on $\mathbf{CP}^3$, an observation initially made by Ward [20] and described in detail in [3], [1]. From this point, [2] gives a classification of these bundles in terms of the Horrocks construction. The result, as we shall see, is that instantons are described by linear maps between vector spaces which may be interpreted as the kernels of twisted Dirac operators over S^4. An excellent reference for this material is [1].

3.1. The twistor space for S^4. We will identify S^4 with quaternionic projective space, $\mathbf{HP}^1$. Here we consider $\mathbf{H}^2$ as a left $\mathbf{H}$ module so that $(x, y) \sim (qx, qy)$, where $(x, y) \in \mathbf{H}^2 - \{0\}$ and $q \in \mathbf{H} - \{0\}$. Notice that the map

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 + z_2 j, z_3 + z_4 j) \tag{3.1}$$

gives an isomorphism of complex vector spaces $\mathbf{C}^4 \cong \mathbf{H}^2$. This gives the fiber bundle

$$\begin{tikzcd} \mathbf{CP}^1 \arrow[r, hook] & \mathbf{CP}^3 \arrow[d, "p"] \\ & S^4 \cong \mathbf{HP}^1, \end{tikzcd} \tag{3.2}$$

where a complex line in $\mathbf{C}^4$ is mapped into the quaternionic line in which it is contained. This demonstrates $\mathbf{CP}^3$ as the so-called Penrose twistor space for S^4 described in [3]. Left quaternionic multiplication by j induces a bundle map covering the identity in $\mathbf{HP}^1$ which will be denoted by $j: \mathbf{CP}^3 \to \mathbf{CP}^3$. In terms of homogeneous coordinates, j is written

$$j(z_1, z_2, z_3, z_4) = (-\overline{z}_2, \overline{z}_1, -\overline{z}_3, \overline{z}_4).$$

This map is antiholomorphic and induces the antipodal map on the fibers of (3.2).

One should think of a point $z \in \mathbf{CP}^3$ as describing a point $p(z) \in S^4$ and a complex structure for a Euclidean neighborhood of $p(z)$. This points to the following correspondence of Atiyah and Ward (see [1]).

Theorem 3.1. *There is a natural* 1-1 *correspondence between gauge equivalence classes of anti-self-dual connections on an* $SU(2)$*-bundle* E *over* S^4 *and isomorphism classes of holomorphic structures on* $\widetilde{E} = p^*(E)$ *over* $\mathbf{CP}^3$, *trivial on the fibers of* (3.2), *possessing a holomorphic nondegenerate skew form* $(\ ,\ )$ *and an antiholomorphic bundle map* $\sigma: \widetilde{E} \to \widetilde{E}$ *covering* j *on* $\mathbf{CP}^3$, *and satisfying* $\sigma^2 = -1$ *and* $(\sigma u, \sigma v) = \overline{(u, v)}$.

3.2. The Horrocks construction and ADHM classification. Using the identification of anti-self-dual $SU(2)$-connections over S^4 with holomorphic bundles over $\mathbf{CP}^3$ described in the preceding section, [2] shows how to construct all instantons by using the Horrocks construction of algebraic bundles over $\mathbf{CP}^3$.

Let W and V be complex vector spaces of complex dimension k and $2k + 2$, respectively. In addition, assume that W has a real structure, i.e., an antilinear map $\sigma: W \to W$ so that $\sigma^2 = +1$, and that V has a quaternionic structure which we will also denote by σ so that $\sigma: V \to V$ and $\sigma^2 = -1$. Furthermore, suppose V is endowed with a positive definite Hermitian inner product $\langle\ ,\ \rangle$. Notice that this results in a nondegenerate skew form on V defined by $(u, v) = \langle u, \sigma v\rangle$. For a subspace $U \subset V$, denote its annihilator under the skew form by U^o; that is, $U^o = \{v \in V | (u, v) = 0\}$.

For $z = (z_1, z_2, z_3, z_4) \in \mathbf{C}^4$, consider a map $A(z): W \to V$ which is complex linear for a fixed value of $z \in \mathbf{C}^4$ and depends linearly on z. In other words, $A(z)$ has the form $A(z) = \sum_{i=1}^4 A_i z_i$, where A_i are linear maps from W to V.

Let $U_z = A(z)W \subset V$ and suppose that $A(z)$ satisfies the further requirements:

Nondegeneracy. For all $z \in \mathbf{C}^4 - \{0\}$, U_z is a k-dimensional subspace of V. This implies that $A(z)$ has maximal rank for all z.

Isotropy. For all $z \in \mathbf{C}^4 - \{0\}$, U_z is isotropic with respect to the skew form on V, i.e.,

$$U_z \subset U_z^o. \tag{3.3}$$

H *structure.* $A(z)$ commutes with the structure maps in the following fashion:

$$\sigma\{A(z)(w)\} = A(jz)(\sigma w) \quad \text{for all } z \in \mathbf{C}^4 \text{ and } w \in W. \tag{3.4}$$

This will describe a holomorphic bundle over $\mathbf{CP}^3$ as follows. For $[z] \in \mathbf{CP}^3$, define $\widetilde{E}_{[z]} = U_z^o/U_z$. Then $\widetilde{E}_{[z]}$ is a two-dimensional complex vector space varying algebraically over $\mathbf{CP}^3$ and hence defines an algebraic two-plane bundle over $\mathbf{CP}^3$. By Theorem 3.1, this defines an instanton.

Among all such maps, call $A(z)$ equivalent to $A'(z)$ if there are isomorphisms $\psi_W: W \to W'$ and $\psi_V: V \to V'$ commuting with the structure maps and preserving the skew forms so that the following diagram commutes for all z:

$$\begin{tikzcd} W \arrow[r, "A(z)"] \arrow[d, "\psi_W"'] & V \arrow[d, "\psi_V"'] \\ W' \arrow[r, "A'(z)"] & V' \end{tikzcd}$$

The following is due to Atiyah, Hitchin, Drinfeld, and Manin [2].

Theorem 3.2. *There is a natural* 1-1 *correspondence between gauge equivalence classes of anti-self-dual* $SU(2)$*-connections on the* $SU(2)$

bundle over S^4 with second Chern class k and equivalence classes of maps $A(z)$ as described above.

Furthermore, Hitchin [14] shows that the vector spaces W and V have natural interpretations as the solutions to differential equations. In particular,

$$W \cong \operatorname{Ker}\left(D_E^- : \Gamma\left(S^- \otimes E\right) \to \Gamma\left(S^+ \otimes E\right)\right)$$

and

$$V \cong \operatorname{Ker}\left(D_{S^- \otimes E}^- : \Gamma\left(S^- \otimes S^- \otimes E\right) \to \Gamma\left(S^+ \otimes S^- \otimes E\right)\right).$$

It is important to note that the map $A(z)$ arises naturally in this interpretation. In addition, the cokernels of these Dirac operators vanish since the scalar curvature of S^4 is positive (see [14]).

3.3. $\mathbf{Z}/2p\mathbf{Z}$ invariant instantons. Suppose that $E = (k, m, m')$ is a $\mathbf{Z}/2p\mathbf{Z}$-equivariant $SU(2)$-bundle over S^4 with an invariant ASD $SU(2)$-connection ∇ which is described by the linear map $A\colon W \otimes C^4 \to V$. The naturality of the ADHM construction shows that W and V are, respectively, real and quaternionic $\mathbf{Z}/2p\mathbf{Z}$ representations.

Under the identification $S^4 \cong \mathbf{HP}^1$, the action on S^4 is written

$$\zeta([x : y]) = \left[x\zeta^{1-q} : y\zeta^{-1-q}\right].$$

This lifts to the following action on $\mathbf{CP}^3$:

$$(3.5) \qquad \zeta([z_1 : z_2 : z_3 : z_4]) = \left[z_1\zeta^{1-q} : z_2\zeta^{-1+q} : z_3\zeta^{-1-q} : z_4\zeta^{1+q}\right]$$

which arises from a linear action on $\mathbf{C}^4$.

Due to the naturality of the ADHM construction, a $\mathbf{Z}/2p\mathbf{Z}$ invariant instanton is described by a map $A\colon W \otimes \mathbf{C}^4 \to V$ which is $\mathbf{Z}/2p\mathbf{Z}$-equivariant with respect to these actions. In the next section, we will compute the action on W and V.

4. Index computations

Throughout this section, suppose that $E = (k, m, m')$ is a $\mathbf{Z}/2p\mathbf{Z}$-equivariant $SU(2)$-bundle over S^4 with an invariant ASD $SU(2)$-connection ∇ which is described by the linear map $A\colon W \otimes \mathbf{C}^4 \to V$. In this section, we will compute the action on W and V through the equivariant Atiyah-Singer index theorem.

4.1. The $\mathbf{Z}/2p\mathbf{Z}$ index of W. Recall that we have identified W with the kernel of the twisted Dirac operator

$$D_E^-: \Gamma\left(S^- \otimes E\right) \to \Gamma\left(S^+ \otimes E\right).$$

Write W as a $\mathbf{Z}/2p\mathbf{Z}$ representation

$$W = \sum_{l=0}^{2p-1} w_l \chi_l,$$

where χ_l is the one-dimensional irreducible representation of $\mathbf{Z}/2p\mathbf{Z}$ of weight l. Since the cokernel of the operator D_E^- vanishes, the values of w_l are available through the equivariant Atiyah-Singer index theorem [4]. Recall that this theorem allows the calculation of the Lefschetz numbers $\mathrm{Lef}(D_E^-, \zeta^j)$, where $\zeta = e^{\pi\sqrt{-1}/p}$; we abbreviate this to $\mathrm{Lef}_W(j)$. These Lefschetz numbers are related to the w_l by the Fourier transform

$$w_l = \frac{1}{2p} \sum_{j=0}^{2p-1} \mathrm{Lef}_W(j) e^{-\pi\sqrt{-1}jl/p} \tag{4.1}$$

or

$$\mathrm{Lef}_W(j) = \sum_{l=0}^{2p-1} w_l e^{\pi\sqrt{-1}jl/p}. \tag{4.2}$$

Before computing, notice that W is an actual representation so that $w_l \geq 0$ for all l. We record this in the following crucial lemma.

Lemma 4.1. *A necessary condition for a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle to support an invariant ASD connection is that $w_l \geq 0$ for all l.*

We now begin the index computation. Since our fixed point set simply consists of the two points 0 and ∞, we apply the formula of [4]:

$$\begin{aligned} &\mathrm{Lef}_W(0) = \mathrm{index}\, D_E^- = -\mathrm{ch}(E)\widehat{\mathscr{A}}\left(S^4\right)\left[S^4\right], \\ &\mathrm{Lef}_W(p) = (-1)^{\varepsilon}\,\mathrm{Lef}_W(0), \\ &\mathrm{Lef}_W(j) = \sum_P \nu_j(P) \quad \text{for } j \neq 0, p, \end{aligned} \tag{4.3}$$

where the sum is over the fixed points P and

$$\nu_j(P) = \frac{\mathrm{Tr}\left(\zeta^j|_{E_P \otimes S_P^+}\right) - \mathrm{Tr}\left(\zeta^j|_{E_P \otimes S_P^-}\right)}{\det\left(\left(1-\zeta^j\right)|_{N_P}\right)}. \tag{4.4}$$

In these expressions, ch is the usual Chern character, $\widehat{\mathscr{A}}$ is the $\widehat{\mathscr{A}}$ genus of S^4, and N is the tangent space of P in S^4. Furthermore, $[S^4]$ denotes the orientation class in the top dimensional homology group of S^4. It is important to understand the action of $p \in \mathbf{Z}/2p\mathbf{Z}$ on $S^- \otimes E$. Since it fixes $SU(2)$-connections on S^- and E, the action must be ± 1. For now, define $\varepsilon \in \{0, 1\}$ so that p acts on $S^- \otimes E$ as $(-1)^\varepsilon$. In particular, p acts as $(-1)^m$ on E and as $(-1)^{q+1}$ on $S^\pm$. Hence, $\varepsilon \equiv m + q + 1 \bmod 2$.

Let α denote the generator for $H^4(S^4)$ so that $\alpha([S^4]) = 1$. We compute that $\widehat{\mathscr{A}}(S^4) = 1$ and $\mathrm{ch}(E) = 2 - k\alpha$, where $c_2(E) = k$. Then $\mathrm{Lef}_W(0) = -(2 - k\alpha)[S^4] = k$ and $\mathrm{Lef}_W(p) = (-1)^\varepsilon k$.

For $j \neq 0, p$, we compute $\nu_j(0)$. From (4.4), it follows that

$$\begin{aligned} \nu_j(0) &= \frac{\cos\frac{\pi m j}{p}\left(\cos\frac{\pi(q+1)j}{p} - \cos\frac{\pi(q-1)j}{p}\right)}{4\sin^2\frac{\pi j}{p}\sin^2\frac{\pi q j}{p}} \\ &= \frac{-\cos\frac{\pi m j}{p}}{2\sin\frac{\pi j}{p}\sin\frac{\pi j q}{p}}. \end{aligned}$$

Similarly, we have

$$\nu_j(\infty) = \frac{\cos\frac{\pi m' j}{p}}{2\sin\frac{\pi j}{p}\sin\frac{\pi j q}{p}}.$$

Then using $(m, m') \sim (aq - b, aq + b)$, we obtain for $j \neq 0, p$,

$$\mathrm{Lef}_W(j) = \frac{\cos\frac{\pi j m}{p} - \cos\frac{\pi j m'}{p}}{2\sin\frac{\pi j}{p}\sin\frac{\pi j q}{p}} \tag{4.5}$$

$$= \frac{\sin\frac{\pi j b}{p}\sin\frac{\pi j a q}{p}}{\sin\frac{\pi j}{p}\sin\frac{\pi j q}{p}}. \tag{4.6}$$

4.2. Parity considerations. Parity arguments force half of the w_l to vanish. Recall that $\varepsilon \equiv m + q + 1 \bmod 2$.

Proposition 4.2. $w_l = 0$ *if* $l \not\equiv \varepsilon \bmod 2$.

Proof. We have $\mathrm{Lef}_W(p) = (-1)^\varepsilon\,\mathrm{Lef}_W(0)$. More generally,

$$\begin{aligned} \mathrm{Lef}_W(p + j) &= \frac{\cos\frac{\pi(p+j)m}{p} - \cos\frac{\pi(p+j)m'}{p}}{2\sin\frac{\pi(p+j)}{p}\sin\frac{\pi(p+j)q}{p}} \\ &= (-1)^\varepsilon\,\mathrm{Lef}_W(j). \end{aligned}$$

Then, if $l \not\equiv \varepsilon \bmod 2$,

$$w_l = \frac{1}{2p} \sum_{j=0}^{2p-1} \mathrm{Lef}_W(j)\, e^{-\pi\sqrt{-1}jl/p}$$
$$= \frac{1}{2p} \sum_{j=0}^{p-1} \left[1 + (-1)^{\varepsilon - l}\right] \mathrm{Lef}_W(j)\, e^{-\pi\sqrt{-1}jl/p} = 0. \quad \text{q.e.d.}$$

Notice that for $l \equiv \varepsilon \bmod 2$, we have

$$w_l = \frac{1}{p} \sum_{j=0}^{p-1} \mathrm{Lef}_W(j)\, e^{-\pi\sqrt{-1}jl/p}. \tag{4.7}$$

From here, we will demonstrate the topological constraint on actions on the bundle mentioned in §2.3. This gives an index theoretic proof of the following result of [10].

Proposition 4.3. $ab \equiv k \bmod p$.

Proof. By an easy induction on n, we obtain

$$\frac{\sin n\theta}{\sin\theta} = \sum_{r=1}^{n} e^{\sqrt{-1}[n+1-2r]\theta}.$$

By letting $l(r, s) = aq + b + q + 1 - 2(rq + s)$, (4.6) gives

$$\mathrm{Lef}_W(j) = \frac{\sin\frac{\pi aqj}{p} \sin\frac{\pi bj}{p}}{\sin\frac{\pi qj}{p} \sin\frac{\pi j}{p}} = \sum_{r=1}^{a} \sum_{s=1}^{b} e^{\sqrt{-1}\pi l(r,s)j/p}.$$

Notice that $l(r, s) \equiv \varepsilon \bmod 2$. If $l \equiv \varepsilon \bmod 2$, then (4.7) becomes

$$w_l = \frac{k}{p} + \frac{1}{p} \sum_{j=1}^{p-1} \sum_{r=1}^{a} \sum_{s=1}^{b} e^{\sqrt{-1}\pi[l(r,s)-l]/p}$$
$$= \frac{k - ab}{p} + \frac{1}{p} \sum_{j=0}^{p-1} \sum_{r=1}^{a} \sum_{s=1}^{b} e^{\sqrt{-1}\pi[l(r,s)-l]j/p}.$$

A simple orthogonality relation states that

$$\frac{1}{p} \sum_{j=0}^{p-1} e^{\sqrt{-1}[l(r,s)-l]/p} = \begin{cases} 1 & \text{if } l \equiv l(r, s) \bmod 2p, \\ 0 & \text{otherwise.} \end{cases}$$

Then since w_l is an integer, $(k - ab)/p$ is also an integer. This implies that $ab \equiv k \bmod p$ and completes the proof of the proposition. q.e.d.

4.3. The $\mathbf{Z}/2p\mathbf{Z}$ index of V. As noted previously, V is identified with the kernel of the Dirac operator

$$D^-_{S^- \otimes E} : \Gamma\left(S^- \otimes S^- \otimes E\right) \to \Gamma\left(S^+ \otimes S^- \otimes E\right),$$

and we write $V = \sum_{l=0}^{2p-1} v_l \chi_l$. The action of $p \in \mathbf{Z}/2p\mathbf{Z}$ on the bundle $S^- \otimes S^- \otimes E$ is $(-1)^m$ so that

$$\operatorname{Lef}_V(0) = -\operatorname{ch}\left(S^-\right)\operatorname{ch}(E)\widehat{\mathscr{A}}\left(S^4\right)\left[S^4\right] = 2k+2$$

and

$$\operatorname{Lef}_V(p) = (-1)^m(2k+2).$$

For $j \neq 0$, p, computing as before gives

$$\begin{aligned}\operatorname{Lef}_V(j) = &\left(e^{\pi\sqrt{-1}j(q-1)/p} + e^{\pi\sqrt{-1}j(q-1)/p}\right)\operatorname{Lef}_W(j)\\ &+ e^{\pi\sqrt{-1}jm'/p} + e^{\pi\sqrt{-1}jm'/p}\end{aligned} \tag{4.8}$$

or

$$\begin{aligned}\operatorname{Lef}_V(j) = &\left(e^{\pi\sqrt{-1}j(q+1)/p} + e^{\pi\sqrt{-1}j(q+1)/p}\right)\operatorname{Lef}_W(j)\\ &+ e^{\pi\sqrt{-1}jm/p} + e^{\pi\sqrt{-1}jm/p}.\end{aligned} \tag{4.9}$$

4.4. The $\mathbf{Z}/2p\mathbf{Z}$ index of the deformation complex. The tangent space to the moduli space at an ASD connection $T_{[\nabla]}\mathscr{M}$ is given by the kernel of the operator

$$D^-_{S^+\otimes E}\colon \Gamma\left(S^-\otimes S^+\otimes \eta_E\right) \to \Gamma\left(S^+\otimes S^+\otimes \eta_E\right),$$

where η_E denotes the complexified adjoint bundle of E. At an invariant connection, the weight 0 subspace of $T_{[\nabla]}\mathscr{M}$ gives the tangent space to the invariant moduli space $\mathscr{M}(E) = \mathscr{M}(k, m, m')$. A computation shows that

$$\begin{aligned}\dim\mathscr{M}(E) &= \frac{1}{p}\sum_{j=0}^{p-1}\operatorname{Lef}_{S^+\otimes\eta_E}(j)\\ &= \frac{8k}{p} - 3 + n\\ &\quad + \frac{2}{p}\sum_{j=1}^{p-1}\cot\frac{\pi j}{p}\cot\frac{\pi jq}{p}\left(\sin^2\frac{\pi jm'}{p} - \sin^2\frac{\pi jm}{p}\right),\end{aligned}$$

where $n \in \{0, 1, 2\}$ is the number of m, $m' \not\equiv 0, p$. This may also be obtained from the formula of [11]. One may see that this dimension is always odd.

5. Obstructions to the existence of invariant instantons

In this section, we will develop a necessary condition for a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle to support a $\mathbf{Z}/2p\mathbf{Z}$-invariant instanton. In §6, a construction will show that $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundles satisfying this condition support invariant instantons.

5.1. $\widetilde{S^1}$-equivariant bundles and composites. Recall that an $SU(2)$-bundle E over S^4 is described by three pieces of information: the second Chern class k, and the weights of the $\mathbf{Z}/2p\mathbf{Z}$ action over the fixed points 0 and ∞; that is, (k, m, m'). Furthermore, any solution (a, b) of (2.5) must satisfy $ab \equiv k \bmod p$.

First, consider the $\mathbf{Z}/p\mathbf{Z}$ action on S^4. Under the standard inclusion $i: \mathbf{Z}/p\mathbf{Z} \to S^1$, this action extends to an S^1 action on S^4 as

$$e^{\sqrt{-1}\theta}x = e^{\sqrt{-1}(1+q)\theta/2}xe^{\sqrt{-1}(q-1)\theta/2}$$

for $e^{\sqrt{-1}\theta} \in S^1$ and $x \in \mathbf{H}$. Let $\widetilde{S^1}$ denote the connected double cover of S^1. This allows us to construct some $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundles by restricting the $\widetilde{S^1}$ action on a $\widetilde{S^1}$-equivariant bundle. The results of [10] provide a classification of $\widetilde{S^1}$-equivariant bundles:

Lemma 5.1. *The $\mathbf{Z}/2p\mathbf{Z}$ action on E extends to an $\widetilde{S^1}$ action on E covering the S^1 action on S^4 if and only if there is a solution to* (2.5) *such that $ab = k$.*

Definition. A $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle (k, m, m') is a composite of $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundles $\{(k_i, m_i, m_i')\}_{i=1}^n$ if $k_i > 0$, $k = \sum_{i=1}^n k_i$, $m \equiv m_1 \bmod 2p$, $m_i' \equiv m_{i+1} \bmod 2p$, and $m' \equiv m_n'$.

Our main result can now be stated in terms of composites of $\widetilde{S^1}$-equivariant bundles.

Theorem 5.2. *A $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle supports an invariant ASD connection if and only if it is a composite of $\widetilde{S^1}$-equivariant bundles.*

A few remarks are in order. First, our definition implies that an $\widetilde{S^1}$-equivariant bundle is itself a composite of $\widetilde{S^1}$-equivariant bundles. Also, since $\widetilde{S^1}$-equivariant bundles are well understood by Lemma 5.1, this theorem gives a classification of $SO(3)$-instantons over $L(p, q) \times \mathbf{R}$. Examples show that a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle may be written as a composite in a number of inequivalent ways.

Example 1. Consider the Lens space $L(5, 2)$ and the $\mathbf{Z}/10\mathbf{Z}$-equivariant bundle $(3, 1, 5)$. This is an $\widetilde{S^1}$-equivariant bundle since $a = 1$, $b = 3$ is a solution to (2.5). However, it is also the composite of $(1, 1, 3)$ and $(2, 3, 5)$, each of which is an $\widetilde{S^1}$-equivariant bundle.

Example 2. The $\mathbf{Z}/34\mathbf{Z}$-equivariant bundle $(7, 0, 4)$ over the Lens space $L(17, 3)$ is the composite of $(3, 0, 6)$ and $(4, 6, 4)$. Another decomposition is $(6, 0, 2)$ and $(1, 2, 4)$. This bundle is not an $\widetilde{S^1}$-equivariant bundle.

Example 3. Over the Lens space $L(7, 2)$, the $\mathbf{Z}/14\mathbf{Z}$-equivariant bundle $(1, 0, 8)$ is neither an $\widetilde{S^1}$-equivariant bundle nor a composite of $\widetilde{S^1}$-equivariant bundles. As such, it will support no invariant instanton by Theorem 5.2.

5.2. A difference equation for w_l. To prove Theorem 5.2, we start by showing that if a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle E supports an invariant instanton, then E is a composite of $\widetilde{S^1}$-equivariant bundles. Lemma 4.1 and a difference equation for the w_l will assist us.

First, we consider the special bundles (cp, m, m). In this case,

$$w_l = \begin{cases} c & \text{if } l \equiv \varepsilon \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

For positive c, this is a bundle satisfying $w_l \geq 0$. One easily sees that such a bundle is a composite of $\widetilde{S^1}$-equivariant bundles. We will state this as a lemma now.

Lemma 5.3. *If $E = (cp, m, m)$ with $c > 0$, then $w_l(E) \geq 0$ and E is a composite of $\widetilde{S^1}$-equivariant bundles.* q.e.d.

For this reason, we will assume that $m \equiv m'$ in what follows. Also, because half of the w_l vanish by parity restrictions, we will only consider those w_l with $l \equiv \varepsilon \bmod 2$.

We begin by recalling the Lefschetz numbers of the operator D_E^-. When $j \neq 0, p$, (4.6) implies

$$\text{(5.1)} \qquad \operatorname{Lef}_W(j) = \frac{\sin\frac{\pi aqj}{p}\sin\frac{\pi bj}{p}}{\sin\frac{\pi qj}{p}\sin\frac{\pi j}{p}} = \frac{\left(\zeta^{aqj} - \zeta^{-aqj}\right)\left(\zeta^{bj} - \zeta^{-bj}\right)}{\left(\zeta^{qj} - \zeta^{-qj}\right)\left(\zeta^{j} - \zeta^{-j}\right)}.$$

To express a difference equation for the w_l, define $\Delta_l = w_{l+2q} - w_l$. Then (5.1) together with the Fourier relation for w_l gives

$$\begin{aligned} \Delta_l &= \frac{1}{2p}\sum_{j=0}^{2p-1} \frac{\left(\zeta^{aqj} - \zeta^{aqj}\right)\left(\zeta^{bj} - \zeta^{-bj}\right)}{\left(\zeta^{qj} - \zeta^{-qj}\right)\left(\zeta^{j} - \zeta^{-j}\right)}\left(\zeta^{-2q} - 1\right)\zeta^{-l} \\ &= -\frac{1}{2p}\sum_{j=0}^{2p-1}\left(\frac{\zeta^{aqj} - \zeta^{-aqj}}{\zeta^{q}}\right)\left(\frac{\zeta^{bj} - \zeta^{-bj}}{\zeta^{j} - \zeta^{-j}}\right)\zeta^{-l}. \end{aligned}$$

Using the fact

$$\frac{\zeta^{bj} - \zeta^{-bj}}{\zeta^{j} - \zeta^{-j}} = \sum_{r=1}^{b} \zeta^{(b+1-2r)j},$$

we obtain our difference equation

$$\Delta_l = -\frac{1}{2p} \sum_{j=0}^{2p-1} \left(\zeta^{(a-1)qj} - \zeta^{-(a+1)qj}\right) \sum_{r=1}^{b} \zeta^{(b+1-2r)j} \zeta^{-l}.$$

We will adopt a convention to compute the Δ_l. By (2.4), we may assume that m, m' are represented by integers satisfying $0 \leq m < m' \leq p$. This then gives

$$\Delta_l = \begin{cases} -1 & \text{for } l + q + 1 \in \left\{m+2, m+4, \cdots, m'\right\}, \\ +1 & \text{for } l + 1 + 1 \in \left(-m'+2, -m'+1, \cdots, -m\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

The values for l, of course, are always taken to have the same parity as ε. In the next section, we will use this to prove that a bundle with $w_l \geq 0$ is necessarily a composite of $\widetilde{S^1}$-equivariant bundles.

5.3. Decomposing bundles. The aim of this section is to prove the following

Theorem 5.4. *If E is a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle and $w_l \geq 0$, then E is a composite of $\widetilde{S^1}$-equivariant bundles.*

Because of Lemma 4.1, this implies that a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle supporting an invariant instanton must be a composite of $\widetilde{S^1}$-equivariant bundles. This is one of the implications of Theorem 5.2.

The proof of Theorem 5.4 proceeds by induction on the instanton number k. A series of propositions will give the result. Recall that Lemma 5.3 means that bundles of the form (cp, m, m) satisfy $w_l \geq 0$ and are composites of $\widetilde{S^1}$-equivariant bundles. For this reason, we will assume that $m \not\equiv m'$ in the following. We begin the induction at instanton number $k = 1$.

Proposition 5.5. *If $E = (1, m, m')$ and $w_l(E) \geq 0$, then E is an $\widetilde{S^1}$-equivariant bundle.*

Proof. Since W is a real $\mathbf{Z}/2p\mathbf{Z}$ representation, $w_l = w_{-l}$. Then $w_l = 0$ for all l with the exception that either $w_0 = 1$ or $w_p = 1$. Assume $w_0 = 1$. Then by (4.8), $V = \chi_{\pm(q-1)} + \chi_{\pm m'}$. Also by (4.9), $V = \chi_{\pm(q+1)} + \chi_{\pm m}$. This gives $m = q - 1$ and $m' = q + 1$ for which

$a = 1$ and $b = 1$ are solutions to (2.5). Hence $E = (1, q-1, q+1)$ is an $\widetilde{S^1}$-equivariant bundle. The case $w_p = 1$ is similar.

Proposition 5.6. *Suppose the* $\mathbf{Z}/2p\mathbf{Z}$*-equivariant bundle* $E = (k, m, m')$ *is a composite of the two bundles* $E_1 = (k_1, m_1, m_1')$ *and* $E_2 = (k_2, m_2, m_2')$. *Then*

$$w_l(E) = w_l(E_1) + w_l(E_2) \quad \text{for all } l.$$

Proof. The definition of composite implies that $k = k_1 + k_2$, $m_1 \equiv m$, $m_1' \equiv m_2$, and $m_2' \equiv m'$. Then

$$\begin{aligned}(5.2)\qquad \cos\frac{\pi j m}{p} - \cos\frac{\pi j m'}{p} = \cos\frac{\pi j m_1}{p} - \cos\frac{\pi j m_1'}{p} \\ + \cos\frac{\pi j m_2}{p} - \cos\frac{\pi j m_2'}{p}\end{aligned}$$

for all j. This implies that $\mathrm{Lef}_W(E, j) = \mathrm{Lef}_W(E_1, j) + \mathrm{Lef}_W(E_2, j)$ for all j. Since the Lefschetz numbers and the w_l are related by the Fourier transform, the proposition is proved. q.e.d.

The proof of this proposition is quite easy by the index computation. However, it should be considered as arising from the excision property of the index.

Proposition 5.7. *Given* $E = (k, m, m')$ *with* $w_l(E) \geq 0$, *there is an* $\widetilde{S^1}$*-equivariant bundle* $E_1 = (k_1, m_1, m_1')$ *satisfying the following*:

(1) *Either* $m_1 = m$ *or* $m_1' = m'$.
(2) $0 \leq w_l(E_1) \leq w_l(E)$ *for all* l.

The proof of this proposition is combinatorial and will be given in §5.4. Assuming this proposition for now, we offer a proof of Theorem 5.4.

Proof of Theorem 5.4. Let $E = (k, m, m')$ satisfy $w_l(E) \geq 0$. Assume, by the induction started in Proposition 5.5, that the result holds for all bundles with lower instanton number. Let E_1 be the $\widetilde{S^1}$-equivariant bundle given in Proposition 5.7 and assume for convenience that $m_1 = m$. Then let $E_2 = (k - k_1, m_1', m')$ so that E is the composite of E_1 and E_2. By Proposition 5.6,

$$w_l(E_2) = w_l(E) - w_l(E_1) \geq 0,$$

and E_2 has lower instanton number than E. By induction, E_2 is a composite of $\widetilde{S^1}$-equivariant bundles. Since E_1 is an $\widetilde{S^1}$-equivariant bundle, E is a composite of $\widetilde{S^1}$ equivariant bundles. This completes the proof of Theorem 5.4.

5.4. Proof of Proposition 5.7. We will now begin the proof of Proposition 5.7. Let $E = (k, m, m')$ be a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle satisfying $w_l \geq 0$. We will suppose that the $\mathbf{Z}/2p\mathbf{Z}$ action is in the form used in the last section to derive the difference equations: $0 \leq m < m' \leq p$. For all $l \in \mathbf{Z}/2p\mathbf{Z}$ satisfying $\Delta_l = -1$, let

$$\alpha(l) = \min\left\{j \in \mathbf{Z}^+ | \Delta_{l-2jq} = +1\right\},$$

and set $\alpha = \min\{\alpha(l) | \Delta_l = -1\}$. Let

$$\mathscr{T} = \{|\Delta_l = -1 \text{ and } \alpha(l) = \alpha\}.$$

By letting $b = m' - m$, $\mathscr{T}$ is determined as follows.

Lemma 5.8. *There exists a positive integer β, $0 < \beta \leq b$, so that either one of the following is true:*

(i) $\mathscr{T} = \{m' - (q+1) - 2j | j = 0, 1, \cdots, \beta - 1\}$,

(ii) $\mathscr{T} = \{m - (q+1) + 2j | j = 1, \cdots, \beta\}$.

Proof. Let $l \in \mathscr{T}$ and write $l \equiv m' - (q+1) - 2j$ with $0 \leq j < b$. Also, let i be the smallest positive integer so that $l - 2\alpha q \equiv -m - 2i$. There are two cases to consider. First, suppose that $i > j$. If $j \neq 0$, then $(l+2) - 2\alpha q \equiv -m - 2(i-1)$ so that $(l+2) \in \mathscr{T}$. By inducting downwards on j, it follows that

$$\left\{m' - (q+1), m' - (q+1) - 2, \cdots, l\right\} \subset \mathscr{T},$$

and case (i) holds. Similarly, the case when $i \leq j$ gives (ii). This proves the lemma. q.e.d.

Suppose, for definiteness, that $\mathscr{T} = \{m' - (q+1), \cdots, m' - (q+1) - 2(\beta - 1)\}$. The other case is handled in a similar manner.

Lemma 5.9. $E_1 = (\alpha\beta, m' - 2\beta m')$ *is an* $\widetilde{S^1}$*-equivariant bundle.*

Proof. We need to show that α and β form a solution to equations (2.5) for the bundle with action $(m' - 2\beta, m')$. First, it is clear that β is a solution to

$$2\beta \equiv m' - \left(m' - 2\beta\right) \mod 2p.$$

It only remains to show that $2\alpha q \equiv m' + (m' - 2\beta) \mod 2p$, which is equivalent to showing

$$\left(m' - 2\beta\right) - 2\alpha q \equiv -m' \mod 2p. \tag{5.3}$$

Note that $j \in \mathscr{T}$ if and only if $\Delta_j = -1$ and $j + q + 1 - 2\alpha q \in \{-m' + 2, -m' + 4, \cdots, -m'\}$. Now $m' - 2(\beta - 1) - (q-1) = m' - 2\beta - (q+1) + 2 \in \mathscr{T}$ and $m' - 2\beta - (q+1) \notin \mathscr{T}$. If $\Delta_{m' - 2\beta - (q+1)} = -1$, it must follow that

$m' - 2\beta - 2\alpha q + 2 \equiv -m' + 2$ verifying (5.3). If $\Delta_{m'-2\beta-(q+1)} \neq -1$, then $\beta = b$. In this case,

$$m + 2 - 2\alpha q,\, m + 4 - 2\alpha q,\, \cdots,\, m' - 2\alpha q \in \left\{-m' + 2, \cdots, -m\right\}.$$

Then $m + 2 - 2\alpha q \equiv -m' + 2$ so that $(m' - 2\beta) - 2\alpha q \equiv -m'$ again verifying (5.3). This proves the lemma. q.e.d.

Let w_l' and Δ_l' be defined by the bundle E_1. The difference equation takes the form

$$\Delta_l' = \begin{cases} -1 & \text{if } l + q + 1 \in \left\{m' - \beta + 2, \cdots, m'\right\}, \\ 1 & \text{if } l + q + 1 \in \left\{-m' + 2, \cdots, -m' + \beta\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\Delta_l' \neq 0$ implies that $\Delta_l' = \Delta_l$ and that $\Delta_l = 0$ yields that $\Delta_l' = 0$.

Lemma 5.10. *For $l \equiv j - 2iq$ with $j \in \mathcal{T}$ and $0 < i < \alpha$, $\Delta_l = 0$.*

Proof. Suppose that l is as in the statement of the lemma and that $\Delta_l \neq 0$. First of all, suppose that $\Delta_l = -1$. Then since $\Delta_{j-2\alpha q} = +1$, it follows that $\Delta_{l-2(\alpha-i)q} = +1$. Thus $\alpha(l) \leq \alpha - i < \alpha$, contradicting the minimality of α. If $\Delta_l = +1$, then $\alpha(j) \leq i < \alpha$, again contradicting the choice of α. This completes the proof. q.e.d.

Lemma 5.11.

$$(5.4) \qquad w_l' = \begin{cases} 1 & \textit{if } l \equiv j - 2iq, \textit{ where } j \in \mathcal{T},\ 0 \leq i < \alpha, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Using Lemma 5.10, one simply checks that the difference equation defined by Δ_l' is satisfied and that $\sum w_l' = \alpha\beta$. This proves the lemma.

Lemma 5.12. *For all l, $0 \leq w_1' \leq w_l$.*

Proof. We need only consider the case $l \equiv j - 2iq$, where $j \in \mathcal{T}$ and $0 \leq i < \alpha$, since otherwise $w_l' = 0 \leq w_l$. Since $j \in \mathcal{T}$, it follows that $\Delta_j = w_{j+2q} - w_j = -1 \geq -w_j$ so that $w_j \geq 1 = w_j'$. For $l \equiv j - 2iq$ with $0 < i < \alpha$, Lemma 5.10 gives $\Delta_l = 0$. Then $w_l = w_j \geq w_j' = w_l'$. q.e.d.

We have now demonstrated an $\widetilde{S^1}$-equivariant bundle satisfying the conclusions of Proposition 5.7. This completes the first half of Theorem 5.2 which we state as follows.

Theorem 5.13. *If a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle supports an invariant ASD connection, it is a composite of $\widetilde{S^1}$-equivariant bundles.*

6. Constructing invariant instantons

In this section, we will show the existence of a $\mathbf{Z}/2p\mathbf{Z}$ invariant instanton on a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle that is a composite of $\widetilde{S^1}$-equivariant bundles. Donaldson's theorem relating holomorphic framed bundles on $\mathbf{CP}^2$ to framed instantons on S^4 shows how to construct $\widetilde{S^1}$ invariant solutions on $\widetilde{S^1}$-equivariant bundles. Then we use an equivariant version of Donaldson's grafting result to construct $\mathbf{Z}/2p\mathbf{Z}$ invariant instantons on $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundles.

6.1. The framed moduli space. Let $\widetilde{\mathscr{M}}$ denote the moduli space of framed ASD $SU(2)$-connections on an $SU(2)$-bundle P over S^4. The points of $\widetilde{\mathscr{M}}$ are pairs (∇, f), where ∇ is an ASD $SU(2)$-connection on P and f is a frame for E over ∞; that is, $f: P_\infty \to SU(2)$ is an isomorphism respecting right translation. Two such pairs are equivalent if they differ by a gauge transformation. Neglecting the frame at ∞ gives a map $\widetilde{\mathscr{M}} \to \mathscr{M}$. We will construct a $\mathbf{Z}/2p\mathbf{Z}$ action on $\widetilde{\mathscr{M}}$ that covers the $\mathbf{Z}/2p\mathbf{Z}$ action on $\mathscr{M}$. Then a fixed point in $\widetilde{\mathscr{M}}$ implies a fixed point in $\mathscr{M}$.

Consider $\mathbf{CP}^2$ contained in $\mathbf{CP}^3$ by the following inclusion:

$$(x : y : z) \hookrightarrow (x : y : z : 0).$$

The projection $p: \mathbf{CP}^3 \to S^4$ in (3.2) restricts to a map $p: \mathbf{CP}^2 \to S^4$ that takes the line $\{z = 0\}$ to ∞ and is 1-1 outside this line. Given a framed ASD $SU(2)$-connection on E, pull the bundle, connection, and frame back to $\mathbf{CP}^2$. This gives a holomorphic framed bundle over $\mathbf{CP}^2$ with a fixed trivialization for the bundle over the line $\{z = 0\}$, and gives a map $p^* : \widetilde{\mathscr{M}} \to \mathscr{O}\widetilde{\mathscr{M}}$ where $\mathscr{O}\widetilde{\mathscr{M}}$ denotes the moduli space of holomorphic bundles trivial over the line $\{z = 0\}$ and with a fixed trivialization over this line.

The following is due to Donaldson [8].

Theorem 6.1. *The map p^* is a* 1-1 *correspondence.*

Using this, we will demonstrate a $\mathbf{Z}/2p\mathbf{Z}$ action on $\mathscr{O}\widetilde{\mathscr{M}}$ and hence on $\widetilde{\mathscr{M}}$.

Let $[X]$ denote homogeneous coordinates on $\mathbf{CP}^2$, where $X = (x, y, z) \in \mathbf{C}^3$. Let H, K, L be complex vector spaces of dimensions k, $2k+2$,

k, respectively. A monad is a sequence of linear maps for each $X \in \mathbf{C}^3$,

$$H \xrightarrow{A_X} K \xrightarrow{B_X} L,$$

so that $B_X A_X = 0$ for all X. We also require that A_X is injective and B_X is surjective for all X and that the maps A_X and B_X depend linearly on X; that is,

$$A_X = A_x x + A_y y + A_z z, \qquad B_X = B_x x + B_y y + B_z z.$$

As in the ADHM construction, this defines a holomorphic bundle $\widetilde{E}$ over $\mathbf{CP}^2$ where $\widetilde{E}_{[X]} = \operatorname{Kernel} B_X / \operatorname{Image} A_X$. The condition that the bundle be trivial on the line $\{z = 0\}$ is equivalent to $B_x A_y = B_y A_x$ being an isomorphism. Regard two monads as equivalent if there are isomorphisms so that the following sequence commutes:

$$\begin{array}{ccccc} H & \xrightarrow{A_X} & K & \xrightarrow{B_X} & L \\ \downarrow & & \downarrow & & \downarrow \\ H' & \xrightarrow{A'_X} & K' & \xrightarrow{B'_X} & L' \end{array}$$

We offer now a coordinate description. Denote the $k \times k$ identity matrix by I_k and the $k \times k$ zero matrix by 0_k. Choose bases for H, K, L so that

$$A_x = \begin{pmatrix} I_k \\ 0_k \\ 0 \end{pmatrix}, \qquad A_y = \begin{pmatrix} 0_k \\ I_k \\ 0 \end{pmatrix},$$

$$B_x = \begin{pmatrix} 0_k & I_k & 0 \end{pmatrix}, \qquad B_y = \begin{pmatrix} -I_k & 0_k & 0 \end{pmatrix}.$$

Also, we write

$$A_z = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -\alpha_2 & \alpha_1 & b \end{pmatrix}.$$

The equation $B_z A_z = 0$ is equivalent to $[\alpha_1, \alpha_2] + ba = 0$. Notice that equivalence classes as expressed in (6.1) correspond to orbits under the following action of $GL(k, \mathbf{C})$: for $g \in GL(k, \mathbf{C})$,

$$g\left(\alpha_1, \alpha_2, a, b\right) = \left(g\alpha_1 g^{-1}, g\alpha_2 g^{-1}, ag^{-1}, gb\right).$$

Barth [5] proves

Theorem 6.2. *The space $\mathscr{O}\widetilde{\mathscr{M}}$ of framed holomorphic bundles over $\mathbf{CP}^2$ is given as the quotient, under the $GL(k, \mathbf{C})$ action, of the set of matrices $(\alpha_1, \alpha_2, a, b)$ so that*

(i) $[\alpha_1, \alpha_2] + ba = 0$,

(ii) *for all* $(\lambda, \mu) \in \mathbf{C}^2$,

$$\begin{pmatrix} \alpha_1 + \lambda \\ \alpha_2 + \mu \\ a \end{pmatrix} \quad \textit{is injective}$$

and

$$\begin{pmatrix} -\alpha_1 - \mu & \alpha_1 + \lambda & b \end{pmatrix} \quad \textit{is surjective.}$$

6.2. $\widetilde{S^1}$ invariant instantons on $\widetilde{S^1}$-equivariant bundles. Throughout this section, assume that $E = (k, m, m')$ is an $\widetilde{S^1}$-equivariant bundle with (a, b) a solution to (2.5) so that $ab = k$. This implies that there is an $\widetilde{S^1}$ action on E, where $\widetilde{S^1}$ is the double cover of S^1, with weights $(aq - b, aq + b)$ covering the S^1 action on S^4. In this case, Donaldson's theorem allows us to construct $\widetilde{S^1}$ invariant instantons.

As in (3.5), the $\widetilde{S^1}$ action on S^4 lifts to the following linear action on $\mathbf{CP}^3$:

$$\theta\left([z_1 : z_2 : z_3 : z_4]\right) = \left[z_1\theta^{1-q} : z_2\theta^{-1+q} : z_3\theta^{-1-q} : z_4\theta^{1+q}\right],$$

where $\theta \in \widetilde{S^1}$. Fortuitously, the action restricts to an action on $\mathbf{CP}^2$:

$$\theta\left([x : y : z]\right) = \left[x\theta^{1-q} : y\theta^{-1+q} : z\theta^{-1-q}\right].$$

There is an action of $\widetilde{S^1}$ on $\mathscr{O}\widetilde{\mathscr{M}}$ given by $\theta(A_X, B_X) = (A_{\theta(X)}, B_{\theta(X)})$. By Theorem 6.1, this gives an action in $\widetilde{\mathscr{M}}$. As this action is induced from the action on S^4 lifted to $\mathbf{CP}^2$, it covers the action on $\mathscr{M}$ under the map $\widetilde{\mathscr{M}} \to \mathscr{M}$. We will now construct an $\widetilde{S^1}$ fixed point in $\mathscr{O}\widetilde{\mathscr{M}}$, thus giving a fixed point in $\mathscr{M}$. To do this, we will construct a monad (A_X, B_X) together with an $\widetilde{S^1}$ action on H, K, and L so that the following commutes:

$$(6.2) \qquad \begin{array}{ccccc} H & \xrightarrow{A_X} & K & \xrightarrow{B_X} & L \\ {\scriptstyle\theta}\downarrow & & {\scriptstyle\theta}\downarrow & & {\scriptstyle\theta}\downarrow \\ H & \xrightarrow{A_{\theta(X)}} & K & \xrightarrow{B_{\theta(X)}} & L \end{array}$$

Using W, V from the ADHM construction, identify H and L with W and K with V. The $\mathbf{Z}/2p\mathbf{Z}$ action on W has Lefschetz numbers

$$\mathrm{Lef}_W(j) = \sum_{r=1}^{a}\sum_{s=1}^{b} e^{\pi\sqrt{-1}l(r,s)j/p},$$

where $l(r, s) = aq + b + q + 1 - 2(rq + s)$. Then

$$W = \sum_{r=1}^{a} \sum_{s=1}^{b} \chi_{l(r,s)},$$

and choose a basis for W, $\{e_{i,j} | i = 1, \cdots, a;\ j = 1, \cdots, b\}$, so that $\widetilde{S^1}$ acts on $e_{i,j}$ with weight $l(i, j)$.

Similarly, recall (4.8):

$$\begin{aligned}\mathrm{Lef}_V(j) = &\left(e^{\pi\sqrt{-1}(1-q)j/p} + e^{\pi\sqrt{-1}(-1+q)j/p}\right)\mathrm{Lef}_W(j)\\ &+ e^{\pi\sqrt{-1}jm'/p} + e^{-\pi\sqrt{-1}jm'/p}.\end{aligned}$$

Then

$$V = \sum_{r=1}^{a}\sum_{s=1}^{b}\left(\chi_{l(r,s)-1+q} + \chi_{l(r,s)+1-q}\right) + \chi_{aq+b} + \chi_{-(aq+b)}.$$

Choose a basis for V of the form

$$\left\{f^+_{i,j}, f^-_{i,j}, f_{aq+b}, f_{-(aq+b)} | i = 1, \cdots, a; j = 1, \cdots, b\right\},$$

where $\widetilde{S^1}$ acts on $f^{\pm}_{i,j}$ with weight $l(r, s) \pm (q-1)$ and on $f_{\pm(aq+b)}$ with weight $\pm(aq+b)$.

Define

$$\alpha_1\left(e_{i,j}\right) = \begin{cases} f^-_{i,j+1} & \text{if } i \le j < b,\\ 0 & \text{otherwise},\end{cases} \qquad \alpha_2\left(e_{i,j}\right) = \begin{cases} f^+_{i+1,j} & \text{if } i \le j < a,\\ 0 & \text{otherwise},\end{cases}$$

$$a\left(e_{i,j}\right) = \begin{cases} f_{-(aq+b)} & \text{if } i = a,\ j = b,\\ 0 & \text{otherwise},\end{cases} \qquad \begin{aligned} b\left(f_{aq+b}\right) &= e_{1,1},\\ b\left(f_{-(aq+b)}\right) &= 0.\end{aligned}$$

This defines an equivariant monad in the sense of diagram (6.2). We will now verify that it defines a point in $\widetilde{\mathscr{O}\mathscr{M}}$ by checking the conditions in Theorem 6.2. First, let L be the $b \times b$ lower diagonal matrix

$$L = \begin{pmatrix} 0 & 0 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \dots & 1 & 0\end{pmatrix}$$

and I_b the $b \times b$ identity matrix. Then, as block matrices, α_1 and α_2 have the form

$$\alpha_1 = \begin{array}{c|cccccccccc}
 & e_{1,1} & \cdots & e_{1,b} & e_{2,1} & \cdots & e_{2,b} & \cdots & e_{a,1} & \cdots & e_{a,b} \\
\hline
f^-_{1,1} & & & & & & & & & & \\
\vdots & & L & & & 0 & & \cdots & & 0 & \\
f^-_{1,b} & & & & & & & & & & \\
f^-_{2,1} & & & & & & & & & & \\
\vdots & & 0 & & & L & & \cdots & & 0 & \\
f^-_{2,b} & & & & & & & & & & \\
\vdots & & \vdots & & & \vdots & & \ddots & & \vdots & \\
f^-_{a,1} & & & & & & & & & & \\
\vdots & & 0 & & & 0 & & \cdots & & L & \\
f^-_{a,b} & & & & & & & & & &
\end{array}$$

$$\alpha_2 = \begin{array}{c|cccccccccc}
 & e_{1,1} & \cdots & e_{1,b} & \cdots & e_{a-1,1} & \cdots & e_{a-1,b} & e_{a,1} & \cdots & e_{a,b} \\
\hline
f^+_{1,1} & & & & & & & & & & \\
\vdots & & 0 & & \cdots & & 0 & & & 0 & \\
f^+_{1,b} & & & & & & & & & & \\
f^+_{2,1} & & & & & & & & & & \\
\vdots & & I_b & & \cdots & & 0 & & & 0 & \\
f^+_{2,b} & & & & & & & & & & \\
\vdots & & \vdots & & \ddots & & \vdots & & & \vdots & \\
f^+_{a,1} & & & & & & & & & & \\
\vdots & & 0 & & \cdots & & I_b & & & 0 & \\
f^+_{a,b} & & & & & & & & & &
\end{array}$$

With respect to the same ordering of the basis for W,

$$a = \begin{array}{c|cccc}
f_{-(aq+b)} & 0 & \cdots & 0 & 1 \\
f_{aq+b} & 0 & \cdots & 0 & 0
\end{array}$$

and

$$b = \begin{array}{cc}
f_{-(aq+b)} & f_{aq+b} \\
\hline
0 & 1 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0.
\end{array}$$

From these forms, it follows that $ba = 0$ and $\alpha_1\alpha_2 = \alpha_2\alpha_1$. Then $[\alpha_1, \alpha_2] + ba = 0$ and the nondegeneracy conditions are verified by noticing that there are always nonzero minors. This defines a point in $\widetilde{\mathscr{O}\mathscr{M}}$ that is fixed under the action of $\widetilde{S^1}$ because of the equivariance expressed in (6.2).

As the bundle $\widetilde{E}$ is defined by $\operatorname{Kernel} B_X / \operatorname{Image} A_X$, and the $\widetilde{S^1}$ action commutes with B_X and A_X, this gives the action on the bundle $\widetilde{E}$ covering the action on $\mathbf{CP}^2$. Over the fixed point $[0: 0: 1]$, the action has weights $\pm(aq - b)$ while over $[1: 0: 0]$, the action has weights $\pm(aq + b)$. This action descends to an action on the bundle E over S^4 fixing the connection. This demonstrates the existence of ASD $SU(2)$-connection on E invariant under an $\widetilde{S^1}$ action with weights $(aq - b, aq + b)$ over the fixed points 0 and ∞. This gives a $\mathbf{Z}/2p\mathbf{Z}$ invariant instanton on a $\widetilde{S^1}$-equivariant bundle. Furuta has announced similar results [13].

6.3. Equivariant grafting. For a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle $E = (k, m, m')$, denote the moduli space of $\mathbf{Z}/2p\mathbf{Z}$ invariant ASD $SU(2)$-connections as $\mathscr{M}(E) = \mathscr{M}(k, m, m')$. Suppose that E is a composite of the bundles $E_1 = (k_1, m_1, m_1')$ and $E_2 = (k_2, m_2, m_2')$ with principal bundles P_1 and P_2, respectively, and that $\mathscr{M}(E_1)$ and $\mathscr{M}(E_2)$ are nonempty. The definition of composite implies that $m_1' \equiv m_2$. Then let I be the space of $\mathbf{Z}/2p\mathbf{Z}$-equivariant isomorphisms $P_{1,\infty} \to P_{2,0}$ modulo $\{\pm 1\}$. In particular, if $m_2 \not\equiv -m_2$, then $I \cong S^1$, and if $m_2 \equiv -m_2$, then $I \cong SO(3)$. Braam [6] has proven an equivariant version of Donaldson's grafting result [9] which we now apply.

Theorem 6.3. *Given points $A_i \in \mathscr{M}(E_i)$, there are neighborhoods $A_i \in U_i \subset \mathscr{M}(E_i)$ so that there is a local diffeomorphism*

$$\Phi: U_1 \times U_2 \times I \to \mathscr{M}(E).$$

In particular, $\mathscr{M}(E) \neq \varnothing$.

Combining this with the results of the last section gives the immediate corollary:

Corollary 6.4. *If a $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle is a composite of $\widetilde{S^1}$-equivariant bundles, then there is a $\mathbf{Z}/2p\mathbf{Z}$ invariant instanton.*

Taken together with Theorem 5.13, this completes the proof of Theorem 5.2.

7. Examples

In this section, we will show how to explicitly determine moduli spaces by several examples. First, we derive a set of equations that the equivariant maps $A(z)$ must satisfy. Then we briefly discuss two group actions on S^4 which will give additional information about $\mathscr{M}(k, m, m')$. In particular, we can compute the Euler characteristic of $\mathscr{M}(k, m, m')$ following the technique of Furuta [13]. Finally we compute the moduli space for a few examples.

7.1. Equations for the ADHM matrices. This section is concerned with deriving a coordinate description of the ADHM matrices. We do this by choosing convenient bases for W and V. At this point, we do not require $\mathbf{Z}/2p\mathbf{Z}$-equivariance of the maps $A(z)\colon W \to V$.

Write $A(z) = \sum A_i z_i$. For a choice of the basis for W, write the structure map σ as complex conjugation composed with a matrix J, where $J^2 = I_k$. Then choose a basis for V so that

$$A_i = \begin{pmatrix} I_k \\ 0_k \\ 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0_k \\ I_k \\ 0 \end{pmatrix}.$$

In terms of this basis for V, the structure map σ on V is complex conjugation composed with the linear map

$$\tilde{J} = \begin{pmatrix} 0 & -J & 0 \\ J & 0 & 0 \\ 0 & 0 & h \end{pmatrix},$$

where h is the 2×2 matrix $h = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. With this basis, take the standard Hermitian inner product on V. The skew form is then given by the matrix $\tilde{J}$.

Write

$$A_3 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix}, \qquad A_4 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ b \end{pmatrix}.$$

The isotropy condition given in (3.3) implies that

$$\beta_1 = -J\overline{\alpha_2}J, \qquad \beta_2 = J\overline{\alpha_1}J, \qquad b = h\bar{a}J.$$

Then the quaternionic structure condition, (3.4), gives

$$\alpha_i^t = J\alpha_i J, \tag{7.1}$$

$$[\alpha_1, \alpha_2] + b^* a = 0, \tag{7.2}$$

$$\left[\alpha_1, \alpha_1^*\right] + \left[\alpha_2, \alpha_2^*\right] + b^* b - a^* a = 0. \tag{7.3}$$

These equations, along with the condition that $A(z)$ is injective for every z, characterize the linear maps representing ASD $SU(2)$-connections on S^4. The condition that two sets of matrices (α_1, α_2, a) and $(\alpha_1', \alpha_2', a')$ be gauge equivalent is that there be a $g_1 \in O(k)$, the $k \times k$ real orthogonal group, and $g_2 \in SU(2)$ so that

$$\alpha_i' = g_1^{-1}\alpha_i g_1, \qquad a' = g_2 a g_1.$$

7.2. Further actions on $\mathcal{M}$. Before we explicitly construct some $\mathbf{Z}/2p\mathbf{Z}$ moduli spaces, we will consider two more group actions on $\mathcal{M}$, the moduli space over S^4.

First, there is the action of $\mathbf{R}^+$, the positive reals under multiplication, on S^4 given by dilation. In terms of quaternionic multiplication $\lambda(x) = \lambda x$, where $\lambda \in \mathbf{R}^+$. This is an action by conformal transformations rather than isometries. Since this action commutes with the $\mathbf{Z}/p\mathbf{Z}$ action on S^4, there is an induced action on $\mathcal{M}(k, m, m')$. Denote the quotient moduli space $\mathcal{M}'(k, m, m') = \mathcal{M}(k, m, m')/\mathbf{R}^+$.

Furthermore, let $T^2 = S^1 \times S^1$ act on S^4 by

$$(\zeta_1, \zeta_2)\, x = (\zeta_1\zeta_2)\, x\left(\zeta_1\overline{\zeta_2}\right).$$

Notice that this extends the $\mathbf{Z}/p\mathbf{Z}$ action since $\mathbf{Z}/p\mathbf{Z}$ is included in T^2 by $e^{\sqrt{-1}\pi j/p} \mapsto (e^{\sqrt{-1}\pi j/p}, e^{\sqrt{-1}\pi jq/p})$. As this commutes with the $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{R}^+$ actions on S^4, it defines an action of T^2 on $\mathcal{M}(k, m, m')$ and $\mathcal{M}'(k, m, m')$. Consider the inclusion $S^1 \hookrightarrow T^2$ given by $\zeta \mapsto (\zeta, \zeta^n)$ for some n. For integers a, b so that $ab = k$, the construction in §6 gives an instanton gauge invariant under the S^1 action on $\mathcal{M}$ and invariant under the $\widetilde{S^1}$ action on the $SU(2)$-bundle with weights $(an - b, an + b)$. Furthermore, this construction shows that if $n > b$, the fixed point set in this moduli space is simply an arc. Indeed, this arc is fixed by the full T^2 action. This implies that the T^2 action on $\mathcal{M}'(k, m, m')$ has a fixed point for every solution a, b to (2.5) so that $ab = k$. Conversely, every fixed point arises in this way. An application of the Lefschetz fixed point theorem shows that

Proposition 7.1. *$\chi(\mathcal{M}(E)) =$ the number of solutions to* (2.5) *such that $ab = k$. In particular, $\chi(\mathcal{M}(E)) \geq 0$.*

This idea comes from Furuta [13].

A number theoretic computation shows that

Proposition 7.2. *If $\dim \mathcal{M}(E) = 1$, then $\mathcal{M}(E) = \mathbf{R}$.*

Proposition 7.3. *If $\dim \mathcal{M}(E) = 3$, then $\mathcal{M}'(E) = S^2 - B$, where B is a collection of either 0, 1, or 2 points.*

Proof. It is well known that these moduli spaces are orientable. Moreover, if $\mathscr{M}'(E)$ is compact, then Theorems 5.2 and 6.3 imply that E is an $\widetilde{S}^1$-equivariant bundle and hence $\chi(\mathscr{M}'(E)) > 0$. For this reason, the two-torus $S^1 \times S^1$ cannot occur as a reduced moduli space. q.e.d.

The examples that follow show that all of the possibilities in Proposition 7.3 are realized.

7.3. Examples.

Example 1. Consider the Lens space $L(5\,,\,1)$ along with the $\mathbf{Z}/2p\mathbf{Z}$-equivariant bundle $(2\,,\,1\,,\,3)$. Notice that this can be written as an $\widetilde{S}^1$-equivariant bundle with $a = 2$, $b = 1$. Another solution to (2.5) is $a = 1$, $b = 2$. By the results of the previous section, we expect the Euler characteristic of $\mathscr{M}(2\,,\,1\,,\,3)$ to be 2.

We will describe instantons on this bundle. Recall that the $\mathbf{Z}/2p\mathbf{Z}$ action on $\mathbf{C}^4$ is written

$$\zeta\,(z_1\,,\,z_2\,,\,z_3\,,\,z_4) = \left(z_1\zeta^{1-q}\,,\,z_2\zeta^{-1+q}\,,\,z_3\zeta^{-1-q}\,,\,z_4\zeta^{1+q}\right).$$

We compute $W = \chi_1 + \chi_9$ and $V = 2\chi_1 + 2\chi_9 + \chi_3 + \chi_7$. Choose bases for W and V as in §6. Then

$$A_3 = \begin{array}{c|cc}
 & 1 & 9 \\
\hline
1 & 0 & 0 \\
9 & \lambda_1 & 0 \\
\hline
1 & 0 & 0 \\
9 & \lambda_2 & 0 \\
\hline
3 & 0 & 0 \\
7 & 0 & \lambda_3
\end{array}$$

where the weights of the actions on W and V are written on the top and side, respectively. In terms of these bases, $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\alpha_1 = \begin{pmatrix} 0 & 0 \\ \lambda_1 & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 0 \\ \lambda_2 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_3 \end{pmatrix}, \quad b = \begin{pmatrix} \overline{\lambda_3} & 0 \\ 0 & 0 \end{pmatrix}.$$

(7.1) and (7.2) are automatically satisfied. Then (7.3) gives

$$|\lambda_1|^2 + |\lambda_2|^2 = |\lambda_3|^2.$$

The nondegeneracy condition is simply that $\lambda_3 \neq 0$. Some of these maps are gauge equivalent. By quotienting out the equivariant gauge group, we obtain $\mathscr{M}(2, 1, 3) \cong \mathbf{CP}^1 \times \mathbf{R}$. As expected, this has Euler characteristic 2. The fixed points under the T^2 action are the points given by $\lambda_1 = 0$ and $\lambda_2 = 0$.

Example 2. On the Lens space $L(5, 2)$, we will construct $\mathscr{M}(3, 1, 5)$. The equivariant maps are described by

$$A_3 = \begin{array}{c|ccc} & 2 & 8 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 7 & 0 & 0 & \lambda_2 \\ 9 & \lambda_1 & 0 & 0 \\ \hline 3 & 0 & 0 & 0 \\ 9 & \lambda_3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 5 & 0 & \lambda_4 & 0 \\ 5 & 0 & 0 & 0 \end{array}$$

Here, J is written as

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using equation (7.1), write

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_1 \\ \lambda_1 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & \lambda_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The nondegeneracy condition is that $\lambda_1 \neq 0$ and $\lambda_4 \neq 0$. (7.3) becomes

$$|\lambda_1|^2 + |\lambda_3|^2 = |\lambda_4|^2.$$

Again, quotienting out the gauge group gives $\mathscr{M}'(3, 1, 5) \cong \mathbf{CP}^1 - \{\infty\}$. Notice that $\mathscr{M}'$ has Euler characteristic 1 corresponding to the solution $a = 1$, $b = 3$ to (2.5). In addition, the noncompactness of $\mathscr{M}'$ arises since $(3, 1, 5)$ is the composite of the bundles $(1, 1, 3)$ and $(2, 3, 5)$. Since $\mathscr{M}(1, 1, 3) \cong \mathscr{M}(2, 3, 5) \cong \mathbf{R}$, Braam's equivariant grafting result implies that there is a local diffeomorphism $\mathbf{R} \times \mathbf{R} \times S^1 \to \mathscr{M}(3, 1, 5)$ which is verified in the model.

Example 3. Take the bundle $(7, 0, 4)$ over the Lens space $L(17, 3)$.

$$A_3 = \begin{array}{c|ccccccc}
 & 0 & 2 & 32 & 8 & 26 & 14 & 20 \\
\hline
32 & 0 & \lambda_9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & \lambda_7 & 0 \\
28 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 \\
16 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8 \\
22 & 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & \lambda_5 & 0 & 0 & 0 \\
30 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Applying (7.1) gives

$$\alpha_1 = \begin{pmatrix}
0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 \end{pmatrix},$$

$$a = \begin{pmatrix} 0 & 0 & 0 & \lambda_5 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(7.2) gives

$$\lambda_1\lambda_3 - \lambda_2\lambda_5 = 0$$

while (7.3) gives

$$|\lambda_1|^2 = |\lambda_3|^2, \quad |\lambda_6|^2 = |\lambda_8|^2, \quad |\lambda_6|^2 = |\lambda_3|^2 + |\lambda_5|^2.$$

The nondegeneracy conditions state that all variables must be nonzero. One sees that $\mathscr{M}(7, 0, 4) \cong (\mathbf{CP}^1 - \{0, \infty\}) \times \mathbf{R}$. Since this bundle is not an $\widetilde{S}^1$-equivariant bundle, the Euler characteristic of $\mathscr{M}$ is zero. The two ends of $\mathscr{M}' \cong \mathbf{CP}^1 - \{0, \infty\}$ correspond to the two decompositions: $(3, 0, 6)$, $(4, 6, 4)$ and $(6, 0, 2)$, $(1, 2, 4)$.

Example 4. Take the bundle $(2, 0, 0)$ over the Lens space $L(2, 1)$. Recall that $L(2, 1) \cong SO(3)$. The equivariant maps are:

$$A_3 = \begin{array}{c|cc} & 0 & 2 \\ \hline 0 & 0 & \lambda_4 \\ 2 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_5 \\ 2 & \lambda_2 & 0 \\ \hline 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \end{array}$$

Equations (7.1) give

$$\alpha_1 = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & \lambda_2 \\ \lambda_2 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & \lambda_3 \\ 0 & 0. \end{pmatrix}$$

Then $\mathscr{M}(2, 0, 0) \cong SO(3) \times \mathbf{R} \times \mathbf{R}$ or $\mathscr{M}' \cong SO(3) \times \mathbf{R}$. This has Euler characteristic zero as expected since it is not an $\widetilde{S}^1$-equivariant bundle.

Notice that the Pontrjagin charge of this bundle is 4. One end of $\mathscr{M}'$ corresponds to a charge popping off at a point of $SO(3)\times\mathbf{R}$ as in Donaldson's original proof [7]. The other end is due to the decomposition $(1, 0, 2)$, $(1, 2, 0)$ and has the form $\mathbf{R}\times\mathbf{R}\times SO(3)$ as predicted from Theorem 6.3.

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